Series Solution to a Second Order Linear Differential Equation with Polynomial Coefficients

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1 Overview

Throughout this paper, we will discuss the general series solution to the second order linear differential equation $a(x)y'' + b(x)y' + c(x)y = 0$, where $a(x)$, $b(x)$, and $c(x)$ are polynomials. A major focus is the construction of a solution about the singular points of this equation; the methods described for evaluating the series solution to the differential equation about ordinary points is no longer effective. This is because the solution of $a(x)y'' + b(x)y' + c(x)y = 0$ at a singular point x_0 is not analytic. In other words, it is not differentiable at a singular point x_0, so it is impossible to construct a power series that represents the solution with powers $(x - x_0)$.

Thus, we must find another way to construct a power series solution to the equation outlined above. As discussed by Boyce and Diprima (2012), one might question whether it is even necessary to consider the singular points of a differential equation since there are usually few of them. This is not ideal, however, since there is often a significant behavioral change of the solution in the neighborhoods of the singular points. In the scope of the real world, as in the model of a physical system, the differential equation generally has the most noteworthy behavior near singular points (Boyce & Diprima, 2012).

Fundamentally, in the real world, almost nothing is perfect (analytic or ordinary) all the time. Almost always, there will be some part of a model that requires further observation due to abnormal behavior (such as at a singular point). Hence, the ability to construct a series solution about a singular point becomes immensely important.

2 Singularities and Regular Singular Points

Let us reconsider the equation $a(x)y'' + b(x)y' + c(x)y = 0$ and rewrite it as

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{b(x)}{a(x)}$ and $Q(x) = \frac{c(x)}{a(x)}$. If either $P(x) \to \infty$ or $Q(x) \to \infty$ as $x \to x_0$, then x_0 is called a *singular point* or a *singularity* (Nagle, Saff, & Snider, 2018). Since the methods of constructing a series about an ordinary point are not effective about singularities, it becomes necessary to seek another method.

To accomplish this feat in a reasonable manner, according to Boyce and Diprima (2012), we must narrow our view to cases in which the singularities of $P(x)$ and $Q(x)$ are not overly severe (Boyce & Diprima, 2012). As a result, we focus on constructing a series about regular singular points: if $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ remain finite (analytic) as $x \to x_0$, then x_0 is a *regular singular point* (Nagle, Saff, & Snider, 2018).

3 Cauchy-Euler Equation

To initiate the method of finding a series solution to $a(x)y'' + b(x)y' + c(x)y = 0$, I will summarize the method of finding the solutions to the Cauchy-Euler Equation

$$x^2y'' + \alpha xy' + \beta y = 0$$

as described by Nagle, Saff, and Snider (2018) and Boyce and Diprima (2012). This is because equations of the form $a(x)y'' + b(x)y' + c(x)y = 0$ tend to behave similarly to Cauchy-Euler equations near regular singular points. As a result, it is useful to note the different types of solutions to the Cauchy-Euler equation for the solution of more general equations.

First, we will consider the scenario for $x > 0$. Let $L[y] = x^2y'' + \alpha xy' + \beta y = 0$, with α and β as real constants. Set $\phi(r, x) := x^r$, and assume that $L[y]$ has a solution of the form $\phi(r, x)$. From this assumption, we obtain

$$\begin{aligned} L[\phi](x) &= x^2(x^r)'' + \alpha x(x^r)' + \beta x^r \\ &= x^2 r(r-1)x^{r-2} + \alpha x r x^{r-1} + \beta x^r \\ &= x^r[r(r-1) + \alpha r + \beta] \end{aligned}$$

Then, $F(r) = r(r-1) + \alpha r + \beta = 0$ is called the *indicial equation*. Now, we must consider three distinct cases for the roots of the indicial equation: (1) real, distinct roots; (2) equal roots; (3) complex conjugate roots.

In case (1), if $L[\phi](x) = 0$ has two real, distinct roots r_1 and r_2 such that $r_1 \neq r_2$ with $L[\phi](x) = (r - r_1)(r - r_2)x^r$, then two unique solutions exist: $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. The solution y then takes the form $y = c_1x^{r_1} + c_2x^{r_2}$.

In case (2), if $r_1 = r_2$, then we are only able to obtain one solution of the form $y_1 = x^{r_1}$. In order to obtain a second solution, we consider the following: since $r_1 = r_2$, $F(r) = (r - r_1)^2$. In this case, $F(r_1)$ and $F'(r_1)$ are both equal to 0. Then, by differentiating $L[\phi](x)$ with respect to r and setting r equal to r_1, we have

$$\begin{aligned} \frac{\partial}{\partial r} L[\phi](x)] &= \frac{\partial}{\partial r}[x^r F(r)] \\ &= \frac{\partial}{\partial r}[x^r (r - r_1)^2] \\ &= x^r ln(x)(r - r_1)^2 + 2(r - r_1)x^r \\ &= 0 \end{aligned}$$

Also, by differentiating interchangeably with respect to x and with respect to r, we have

$$\begin{aligned} \frac{\partial}{\partial r} L[x^r] &= L[\frac{\partial}{\partial r} x^r] \\ &= L[x^r ln(x)] \end{aligned}$$

With this information, we obtain $y_2 = x^{r_1}$. Hence, $y = c_1 x^{r_1} + c_2 x^{r_1} ln(x)$.

In case (3), with $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ with $\mu \neq 0$, recall $x^r = e^{rln(x)}$. Let us consider $x^{\lambda+i\mu}$ since the calculation for the conjugate is almost identical. Then, we have

$$\begin{aligned} x^{\lambda+i\mu} &= e^{(\lambda+i\mu)ln(x)} \\ &= e^{\lambda ln(x)} e^{i\mu ln(x)} \\ &= x^{\lambda}[\cos(\mu ln(x)) + i\sin(\mu ln(x))] \end{aligned}$$

Thus, $y = c_1 x^{\lambda} cos(\mu ln(x)) + c_2 x^{\lambda} sin(\mu ln(x))$.

In our consideration of the scenario for $x < 0$, the solutions for a Cauchy-Euler equation often become complex-valued. However, by making the change of variable $x = -\xi$ with $\xi > 0$, we can obtain real-valued solutions for $x < 0$. Letting $y = u(\xi)$, we have the following: for case (1) as described above, we have $u(\xi) = c_1\xi^{r_1} + c_2\xi^{r_2}$. For case (2), we have $u(\xi) = c_1\xi^{r_1} + c_2\xi^{r_1} ln(\xi)$. For case (3), we have $u(\xi) = c_1\xi^{\lambda} cos(\mu ln(\xi)) + c_2\xi^{\lambda} sin(\mu ln(\xi))$.

Now, overall, we have a general solution set to the Cauchy-Euler equation. Namely, for real and different roots,

$$y = c_1|x|^{r_1} + c_2|x|^{r_2};$$

for real and equal roots,

$$y = c_1|x|^{r_1} + c_2|x|^{r_1} ln(|x|);$$

for complex conjugate roots,

$$y = c_1|x|^{\lambda}\cos(\mu ln(|x|)) + c_2|x|^{\lambda}\sin(\mu ln(|x|)).$$

4 Series Solution About a Regular Singular Point

Now, let us reconsider the general equation $a(x)y'' + b(x)y' + c(x)y = 0$ and rewrite it as we did above in the following way:

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{b(x)}{a(x)}$ and $Q(x) = \frac{c(x)}{a(x)}$. Let us begin constructing a series solution about a regular singular point x_0. For the sake of simplicity, let $x_0 = 0$. Otherwise, the solution can be shown by letting $t = x - x_0$. By the assumption that $x = 0$ is a regular singular point, we have that $\lim_{x \to 0} xP(x)$ and $\lim_{x \to 0} x^2Q(x)$ are finite; thus, they are also analytic at $x = 0$. This implies that $xP(x)$ and $x^2Q(x)$ each have power series expansions:

$$xP(x) = \sum_{n=0}^{\infty} P_n x^n; x^2Q(x) = \sum_{n=0}^{\infty} Q_n x^n.$$

First, consider the simple case where $\lim_{x \to 0} xP(x) = P_0$ and $\lim_{x \to 0} x^2Q(x) = Q_0$. Then, $y'' + P(x)y' + Q(x)y = 0$ can be reduced to the following Cauchy-Euler equation:

$$x^2y'' + P_0xy' + Q_0y = 0.$$

Then, the solutions are similar to the ones discussed in the previous section.

Now, we must consider the less trivial case where the terms of $xP(x)$ and $x^2Q(x)$ are nonzero. Consider the equation $y'' + P(x)y' + Q(x)y = 0$ as described above. In order to visualize our power series in this equation, multiply by x^2. This implies that

$$x^2y'' + x[xP(x)]y' + x^2Q(x)y = 0.$$

We will use the method of Frobenius to evaluate this problem. According to Nagle, Saff, and Snider (2018): "The idea of of the mathematician Frobenius was that since Cauchy-

Euler equations have solutions of the form x^r, then for the regular singular point $x = 0$, there should be solutions...of the form x^r times an analytic function" (Nagle, Saff, & Snider, 2018). This coincides with the discussion of Boyce and Diprima (2012). Hence, we assume that there exist solutions of the form $y = \phi(r,x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$ $(x > 0)$, with $a_0 \neq 0$. Then,

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1};$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting y' and y'' into $x^2y'' + x[xP(x)]y' + x^2Q(x)y = 0$, we obtain the following:

$$x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2} + x \sum_{n=0}^{\infty} P_n x^n \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} Q_n x^n \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$\begin{aligned} &= a_0 F(r) x^r + [a_1 F(r+1) + a_0 P p_1 r + Q_1) x^{r+1}] \\ &\quad + \{a_2 F(r+2) + a_0 (P_2 r + Q_2) + a_1 [P_1 (r+1) + Q_1]\} x^{r+2} \\ &\quad + \ldots + \{a_n F(r+n) + a_0 (P_n r + Q_n) + a_1 [P_{n-1}(r+1) + Q_{n-1}] + \ldots \\ &\quad + a_{n-1}[P_1 (r+n-1) + Q_1]\} x^{r+n} = 0, \end{aligned}$$

where $F(r) = r(r-1) + P_0 r + Q_0$ (similarly to the indicial equation from the previous section). More succinctly, we can say

$$L[\phi](r,x) = a_0 F(r) x^r + \sum_{n=1}^{\infty} \left\{ F(r+n) a_n + \sum_{k=0}^{n-1} a_k [(r+k) P_{n-k} + Q_{n-k}] \right\} x^{r+n} = 0.$$

We now need to find the roots of $F(r)$ (r_1 and r_2); these roots are called the *exponents of the singularity* and will play a role in determining the behavior of the solutions in the neighborhood of the singular point. To find the recurrence relation, we must set the coefficient of x^{r+n} equal to zero. Thus, $F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)P_{n-k} + Q_{n-k}] = 0$ $(n \geq 1)$.

This recurrence relation demonstrates that a_n generally depends on r and its preceding coefficients $a_0, ..., a_{n-1}$. This means that we can compute $a_1, ..., a_n, ...$ in terms of a_0 and the coefficients in the series representations of $xP(x)$ and $x^2Q(x)$; however, $F(r+1), ..., F(r+n), ...$ cannot be equal to zero.

If we have two distinct roots, let $r_1 \geq r_2$. The only values that make the indicial equation $F(r)$ equal to zero are r_1 and r_2. Also, $r_1 \geq r_2$ implies that $r_1 + n \neq r_1$ or r_2 for any $n \geq 1$; now, $F(r_1+1), ..., F(r_1+n), ... \neq 0$ for any $n \geq 1$. Thus, we can find one solution of the form $y = \phi(r, x)$; specifically,

$$y_1(x) = x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right]$$

for $x > 0$, with $a_n(r_1)$ meant to denote that a_n was determined by setting $r = r_1$. Furthermore, we take $a_0 = 0$ to specify the arbitrary constant. Through similar reasoning, moreover, since $r_1 \geq r_2$, $r_2 + n \neq r_1$ for any $n \geq 1$. This implies that there exists a second solution of the same fashion:

$$y_2(x) = x^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right]$$

for $x > 0$.

To cover the scenario for $x < 0$, make the same change of variable $x = -\xi$ with $\xi > 0$. Then, just as in the previous section, we only need to replace x^{r_1} and x^{r_2} with $|x|^{r_1}$ and $|x|^{r_2}$, respectively. The two solutions $y_1(x)$ and $y_2(x)$ must converge for $|x| < \rho$, where both $\sum_{n=0}^{\infty} P_n x^n$ and $\sum_{n=0}^{\infty} Q_n x^n$ converge as well. Within their radii of convergence, our two solutions define functions that are analytic at $x = 0$. Now, any singular behavior is the responsibility of x^{r_1} or x^{r_2}.

Overall, we must realize the following: r_1 and r_2 determine the behavior of the solutions, and they are relatively easy to find; namely, we need only solve the quadratic indicial equation

$$r(r-1)+P_0 r+Q_0=0$$

where P_0 and Q_0 are given by $\lim_{x\to 0} xP(x)$ and $\lim_{x\to 0} x^2Q(x)$. Even though all of the terms of the series representations of $xP(x)$ and $x^2Q(x)$ may not be zero except for P_0 and Q_0, we can use this with the method of Frobenius. Given that we know that the general second order linear differential equation behaves similarly to the Cauchy-Euler about regular singular points, we can use the method that we used for the Cauchy-Euler equation to determine the exponents of the singularity. Then, by the method of Frobenius, we make the assumption that the solution is of the form "Euler coefficients times the power series" to account for the remaining terms. Furthermore, the radii of convergence of our solutions must be at least equal to the distance from the origin to the nearest zero of $a(x)$ other than $x=0$.

Now, we must consider two remaining cases: $r_1=r_2$ or $r_1-r_2=N$ for some some positive integer N. For the case of equal roots, let r be a continuous variable and determine a_n by solving the recurrence relation from above. Based on our choice of a_n for $n\geq 1$, all of the coefficients of $L[\phi](r,x)$ must equal zero, so $L[\phi](r,x)$ reduces to $L[\phi](r,x)=a_0F(r)x^r=a_0(r-r_1)^2x^r$ since r_1 is a repeated root of $F(r)$. If we consider

$$\begin{aligned} L[\frac{\partial\phi}{\partial r}](r,x) &= a_0\frac{\partial}{\partial r}[x^r(r-r_1)^2] \\ &= a_0[x^r ln(x)(r-r_1)^2+2x^r(r-r_1)] \end{aligned}$$

If we set $r=r_1$, then $L[\frac{\partial\phi}{\partial r}](r,x)=0$, just as for the Cauchy-Euler equation. Thus, we can obtain a second solution in the following way: first, evaluate

$$\begin{aligned} \frac{\partial\phi(r,x)}{\partial r} &= \frac{\partial}{\partial r}\left\{x^r\left[a_0+\sum_{n=1}^{\infty}a_n(r)x^n\right]\right\} \\ &= x^r ln(x)\left[a_0+\sum_{n=1}^{\infty}a_n(r)x^n\right]+x^r\sum_{n=1}^{\infty}a_n'(r)x^n \end{aligned}$$

Setting $r = r_1$ yields

$$y_2(x) = y_1(x)ln(x) + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n.$$

The case where $r_1 - r_2 = N$ is beyond the scope of this class, so I will merely state the result. The second solution is defined in the following way:

$$y_2(x) = ay_1(x)ln(x) + x^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right]$$

where c_n and a can be found by substituting the above equation into $x^2y'' + x[xP(x)]y' + x^2Q(x)y = 0$.

5 Example Problem

Consider the differential equation $2x^2y'' - xy' + (1 + x)y = 0$. Here, $a(x) = 2x^2$, $b(x) = -x$, and $c(x) = 1 + x$. Now, we can define $P(x) = \frac{-1}{2x}$ and $Q(x) = \frac{1+x}{2x^2}$. Thus, $x = 0$ is the only singular point. To check if it is a regular singular point, let us evaluate $\lim_{x\to 0} xP(x)$ and $\lim_{x\to 0} x^2Q(x)$.

$$\lim_{x\to 0} xP(x) = \lim_{x\to 0} \frac{-1}{2} = \frac{-1}{2}$$
$$\lim_{x\to 0} x^2Q(x) = \lim_{x\to 0} \frac{1+x}{2} = \frac{1}{2}$$

Since both limits are finite, $x = 0$ is a regular singular point; and we now know that $P_0 = \frac{-1}{2}$, $Q_0 = \frac{1}{2}$, $Q_1 = \frac{1}{2}$, and all of the other $P's$ and $Q's$ are zero. So, the Euler equation corresponding to our initial equation is

$$2x^2y'' - xy' + y = 0.$$

We now assume that there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{r+n}$ so that y' and y'' can be defined as follows:

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$
$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

At this point, we substitute y, y', and y'' into the original equation.

$$2x^2 y'' - xy' + (1+x)y = \sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1) x^{r+n} - \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1}$$

Shifting indices of the summations as needed and rearranging the terms, we obtain the following:

$$a_0[2r(r-1) - r + 1]x^r + \sum_{n=1}^{\infty} \{[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1}\} x^{r+n} = 0$$

In order for the above equation to be satisfied, all of the coefficients of the powers of x must be equal to zero, so the indicial equation must be given by

$$2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (2r-1)(r-1) = 0.$$

So, $r_1 = 1$ and $r_2 = \frac{1}{2}$. These are the exponents of the singularity that will determine the behavior of the solutions around $x = 0$. Now, to obtain the recurrence relation, we must set the coefficient of $x^{r+n} = 0$. In other terms, $[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1} = 0$

for $n \geq 1$. Then, we have

$$a_n = -\frac{a_{n-1}}{2(r+n)^2 - 3(r+n) + 1}$$
$$= -\frac{a_{n-1}}{[(r+n)-1][2(r+n)-1]}$$
$$n \geq 1$$

For each root, we must use the recurrence relation to establish a_n. For $r_1 = 1$,

$$a_n = -\frac{a_{n-1}}{(2n+1)n}$$

for $n \geq 1$. Then,

$$a_1 = -\frac{a_0}{(3)(1)}$$
$$a_2 = -\frac{a_1}{(5)(2)} = \frac{a_0}{(5)(3)(2)(1)}$$
$$a_3 = -\frac{a_2}{(7)(3)} = -\frac{a_0}{(3)(5)(7)(1)(2)(3)}$$

Now, in general, we have

$$a_n = \frac{(-1)^n}{(3)(5)(7)...(2n+1)]n!} a_0$$

for $n \geq 4$. If we multiply a_n above by $\frac{2^n n!}{2^n n!}$, we have

$$a_n = \frac{(-1)^n 2^n}{(2n+1)!} a_0$$

for $n \geq 1$. Omitting the constant multiplier a_0, we have

$$y_1(x) = |x| \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n \right]$$

. We can determine the radius of convergence for $y_1(x)$ via the ratio test; namely,

$$\lim_{n\to\infty} \frac{a_{n+1}x^{n+1}}{a_n x^n} = \lim_{n\to\infty} \frac{2|x|}{(2n+2)(2n+3)} = 0$$

Thus, the radius of convergence $\rho = \infty$, so the series solution converges for all values of x. Moving on to the second root $r_2 = \frac{1}{2}$, we have

$$a_n = -\frac{a_{n-1}}{2n(n-1/2)} = -\frac{a_{n-1}}{n(2n-1)}$$

for $n \geq 1$. Thus,

$$\begin{aligned} a_1 &= -\frac{a_0}{(1)(1)} \\ a_2 &= -\frac{a_1}{(2)(3)} = \frac{a_0}{(1)(3)(1)(2)} \\ a_3 &= -\frac{a_2}{(3)(5)} = -\frac{a_0}{(1)(2)(3)(1)(3)(5)} \end{aligned}$$

Then, in general, we have

$$a_n = \frac{(-1)^n}{[(1)(3)(5)...(2n-1)]n!} a_0$$

for $n \geq 4$. Just as before, multiply a_n by $\frac{2^n n!}{2^n n!}$. Now, we have

$$a_n = \frac{(-1)^n 2^n}{(2n)!} a_0$$

for $n \geq 1$. Omitting the constant multiplier a_0, we have

$$y_2(x) = |x|^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n \right]$$

Also just as above, the radius of convergence $\rho = \infty$, so the series solution converges for all values of x.

6 References

Books:

1. Elementary Differential Equations by Boyce and Diprima

2. Fundamentals of Differential Equations by Nagle, Saff, and Snider

Websites/Online PDFs:

1. http://homepage.divms.uiowa.edu/ idarcy/COURSES/100
   /mainBuchananSingula.pdf

2. http://www.math.mcgill.ca/gantumur/math315w14/downloads/frobenius.pdf

3. http://mathworld.wolfram.com/RegularSingularPoint.html

Note: I did my best to summarize my understanding of this concept from source (1.). I used the other sources to double-check ideas and cross-refernce definitions. Most of this paper, including the example, came from source (1.)