Elliptic Functions and an Introduction to Modular Functions

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1 Abstract

This is a study of modular forms; specifically, the creation of modular forms via the construction of the Weierstrass $\wp$ function. The construction of this function and the creation of modular forms are dependent on a detailed understanding of elliptic functions and their properties.

2 Introduction

The study began with the notion of doubly periodic functions. These are functions with two distinct periods ω_1 and ω_2 whose ratio is not a real number. These two periods are called a fundamental pair of periods, and they generate a lattice $\Omega = (\omega_1, \omega_2)$ whose vertices are linear combinations of both periods. The main concept was then introduced: elliptic functions. A function f is elliptic if it is doubly periodic and meromorphic. In order for f to be a non-constant elliptic function, furthermore, there must be poles or zeros inside of the lattice generated by its fundamental pair of periods. With this information, constructing an elliptic function of order 2 started with considering the principal part of the Laurent expansion near a period ω. This construction resulted in an important function in the case of constructing modular functions, the Weierstrass $\wp$ function, $\wp = \frac{1}{z^2} + \sum_{\omega \neq 0} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}$ where z is a complex variable and ω is a period. This function was manipulated by means of differentiation to construct additional functions related to the coefficients of the derivative $\wp'(z)$. These were defined as the discriminant $\Delta(\tau)$ and Klein's modular function $J(\tau)$. With these functions, unimodular transformations were defined and then generalized into Möbius transformations. Unimodular transformations and Möbius transformations provide initial, more widely-known, examples of modular functions.

According to [5], modular form theory and the study of modular functions are relatively recent ideas. They were developed in conjunction with the theory of elliptic functions in the 19th Century. Modular form theory was then extended largely by German mathematician Erich Hecke in the early to mid 20th Century [6]. As was the case historically, the idea of modular functions in this essay will be constructed with a study of elliptic functions.

This project has spanned the final year of my master's education. I spent the Fall 2020 semester studying elliptic functions under the supervision of Professor Daniel Isaksen. The Winter 2021 semester consisted of synthesizing and summarizing the information that I had studied into a cohesive master's essay. The structure and order of the topics in this essay were guided by the first two chapters in Tom Apostol's thoughtful book *Modular Functions and Dirichlet Series in Number Theory* [1].

3 Doubly periodic functions

In order to effectively construct the idea of elliptic functions, it is necessary to discuss periodic functions.

Definition 3.1. A complex function f is **periodic** with period ω if $f(z+\omega) = f(z)$ whenever z and $z+\omega$ are in the domain of f.

In other words, "a periodic function is a function that repeats its values at regular intervals" [7]. Such functions have been present in mathematics courses up until this point. Consider, for instance, $f(z) = \sin(z)$. Here, $\omega = 2\pi$. The same is true for $f(z) = \cos(z)$. The other trigonometric functions can be defined similarly with different periods, but it is clear that they all repeat their values at regular intervals, a notion with which we are familiar. Now, if we consider a periodic function with two distinct periods ω_1 and ω_2, doubly periodic functions can be defined.

Definition 3.2. A complex function f is **doubly periodic** if it has two periods ω_1 and ω_2 whose ratio $\frac{\omega_2}{\omega_1}$ is not real.

4 Fundamental Pairs of Periods

Definition 4.1. Let f be doubly periodic. The pair of periods (ω_1, ω_2) is referred to as a **fundamental pair** if every period of f is a linear combination of ω_1 and ω_2.

Considering a doubly periodic function is not nearly as nice or easy as considering a periodic function. As it will be seen in the upcoming sections, constructing a non-constant doubly periodic function requires additional requirements and far more deliberation because they are periodic in two dimensions instead of one dimension.

Definition 4.2. The **lattice generated by** ω_1 **and** ω_2, denoted by $\Omega(\omega_1, \omega_2)$, is the set of all $\mathbb{Z}$-linear combinations of ω_1 and ω_2. In the complex plane, this depicts a network of parallelograms whose vertices are all of the linear combinations of ω_1 and ω_2.

Consider a pair of periods $1+i$ and $2i$. If we take linear combinations $\omega = m(1+i) + n(2i)$ for any integers m and n, then the result is indeed a network of parallelograms that span the entire complex plane. Any one parallelogram in this lattice can be referred to as a **period parallelogram**. Refer to Figure 1 for an illustration of the lattice of this pair of periods. The original periods will be labeled in red and their linear combinations will be represented by the other surrounding points. Once connected, they span the entire complex plane.

Theorem 4.1. The two pairs (ω_1, ω_2) and (ω_1', ω_2') are equivalent, meaning $\Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2')$, if and only if there is a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with integer entries and determinant $ad - bc = \pm 1$ such that

$$\begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}.$$

Proof. First, assume that there is a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with determinant ± 1 such that $\begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$. We must show that ω_1 and ω_2 can be expressed as linear combinations of ω_1' and ω_2'. Suppose that both

a and c are non-zero. Now, consider $\omega_2' = a\omega_2 + b\omega_1$ and $\omega_1' = c\omega_2 + d\omega_1$. From here, $\omega_2 = \frac{\omega_2' - b\omega_1}{a}$ and $\omega_2 = \frac{\omega_1' - d\omega_2}{c}$. Thus, $\frac{\omega_2' - b\omega_1}{a} = \frac{\omega_1' - d\omega_1}{c}$, so we have

$$\begin{aligned} c\omega_2' - bc\omega_1 &= a\omega_1' - ad\omega_1 \\ a\omega_1' - c\omega_2' &= ad\omega_1 - bc\omega_1 \\ a\omega_1' - c\omega_2' &= \pm\omega_1 \end{aligned}$$

Similarly, $\omega_1 = \frac{\omega_2' - a\omega_2}{b}$ and $\omega_1 = \frac{\omega_1' - c\omega_2}{d}$. By the same argument as above, ω_2 can be expressed as a linear combination of ω_1' and ω_2' as well. Thus, $\Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2')$.

Now suppose that $a = 0$. Since $ad - bc = \pm 1$, we have that b and c are non-zero. In fact, $b = \pm 1$ and $c = \mp 1$. Therefore, $\omega_1 = \frac{\omega_2'}{b}$ and $\omega_1 = \frac{\omega_1' - c\omega_2}{d}$, so $\omega_1 = \pm\omega_2'$ and $\omega_2 = \omega_1' - d\omega_1 = \omega_1' - d(\pm\omega_2'$. Thus, ω_1 and ω_2 are linear combinations of ω_1' and ω_2', so again $\Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2')$. The case where $c = 0$ is analogous to the case where $a = 0$, and the same result is obtained.

Next, assume that $\Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2')$. With this assumption, consider the following relations:

$$\begin{aligned} \omega_2' &= a\omega_2 + b\omega_1 \\ \omega_1' &= c\omega_2 + d\omega_1 \\ \omega_2 &= m\omega_2' + n\omega_1' \\ \omega_1 &= q\omega_2' + r\omega_1' \end{aligned}$$

where, once again, $a, b, c, d, m, n, q, r \in \mathbb{Z}$. We now have $\begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$ and $\begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} m & n \\ q & r \end{bmatrix} \begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix}$. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$ and $\begin{bmatrix} m & n \\ q & r \end{bmatrix} = B$. Then $\begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = AB \begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix}$. Now, we have

$$\begin{aligned} \omega_2' &= w\omega_2' + x\omega_1' \\ \omega_1' &= y\omega_2' + z\omega_1' \end{aligned}$$

Since ω_1' and ω_2' are linearly independent over $\mathbb{R}$, we must have that $AB = I$ (the identity matrix), so $\det(AB) = \det(A)\det(B) = \det(I)$ Since $\det(I) = 1$, the only way for this equality to hold is if $\det(A) = \pm 1$ and $\det(B) = \pm 1$. We have the desired result, and this completes the proof. $\square$

5 Elliptic functions

Definition 5.1. A function f is **elliptic** if it is doubly periodic and if it is meromorphic (meaning that its only singularities in the finite plane are poles).

Constant functions in the complex plane are trivial examples of elliptic functions. As aforementioned, the construction of non-constant elliptic functions requires a great deal of deliberation, and it will be the focus of the next section. The purpose of this section is to establish some of the fundamental properties of elliptic functions.

Theorem 5.1. A non-constant elliptic function has a fundamental pair of periods.

Proof. Since f is elliptic, the set on which f is analytic is open and connected. Moreover, f must be doubly periodic where the two periods have a non-real ratio. Of all the non-zero periods of f, at least one must be of minimal distance from the origin (if not, f would have arbitrarily small non-zero periods, so it would be constant). Let ω be one such period. Now, choose the period with the smallest non-negative argument (angle with respect to the real axis) with the modulus $|\omega|$ and denote it ω_1. If there are other periods with modulus $|\omega_1|$ other than ω_1 and $-\omega_1$, take the period with the smallest argument greater than the argument of ω_1. Denote it ω_2. If this is not the case, find the next largest circle containing periods that are not equal to $n\omega_1$ for some integer n, and choose a period of the smallest non-negative argument. A period of this nature must exist since f has two non-collinear periods. Denote this period by ω_2, as well. Then, in either case, we have that there are no periods in the triangle created by $0, \omega_1, \omega_2$ other than the vertices. In other words, each period of f must be a $\mathbb{Z}$-linear combination of ω_1 and ω_2, so (ω_1, ω_2) is a fundamental pair of periods. □

Definition 5.2. A **zero** of a complex function f is a complex number z such that $f(z) = 0$. A **pole** of a complex function f is a complex number z such that $\frac{1}{f}(z) = 0$.

Theorem 5.2. If an elliptic function f has no poles in some period parallelogram, then f is constant.

Proof. If a function f has no poles in an arbitrary period parallelogram, then it would be continuous in the period parallelogram. This continuity would allow us to bound the function in the period parallelogram. From periodicity, we have that f is continuous and bounded in the entire complex plane. Thus, by Liouville's Theorem, f is a constant function [8]. □

Theorem 5.3. If an elliptic function f has no zeros in some period parallelogram, then f is constant.

Proof. Consider the reciprocal function $\frac{1}{f}$ and apply Theorem 5.2 to said reciprocal. The result follows in the same way. □

Definition 5.3. Meromorphic functions only have a finite number of poles or zeros. To avoid inconvenient situations where there are zeros or poles on the boundaries of period parallelograms, we can use the fact mentioned before to construct a congruent period parallelogram without poles or zeros on its boundary. This construction will be referred to as a **cell**.

Theorem 5.4. The contour integral of an elliptic function taken along the boundary of any cell is zero.

Proof. If contour integrals are taken along parallel edges, they will cancel due to periodicity. □

Theorem 5.5. The sum of the residues of an elliptic function at its poles in any period parallelogram is zero.

The idea here is to consider the fundamental parallelogram (the parallelogram created by $0, \omega_1, \omega_1, \omega+\omega_2$). Then, we want to consider the contour integral of f about the boundary of this parallelogram. If we orient this parallelogram in a counterclockwise direction and dissect the boundary into four parts (one part for each side of the parallelogram), the parallel sides will nullify each other during integration. Thus, the sum of residues of f will be 0 since said sum is equal to the contour integral of f about the boundary of the period parallelogram [2].

Theorem 5.6. The number of zeros of an elliptic function in any period parallelogram is equal to the number of poles, each counted with multiplicity.

Proposition 5.7. The order of every non-constant elliptic function (the number of zeros or poles in any period parallelogram) must be greater than or equal to 2.

Proof. If a non-constant elliptic function had order 1 and we sum the residues of the poles in a period parallelogram, this sum would be equal to the residue of the single pole. This is contradictory to Theorem 5.5, so the order of any non-constant elliptic function must be at least 2. ☐

6 Construction of elliptic functions

We will be following the construction of Weierstrass. His method involved constructing a elliptic function with a pole of order 2 at the origin $z = 0$ (thus at every period). To begin this construction, consider the Laurent expansion about each period ω:

$$\frac{A}{(z-\omega)^2} + \frac{B}{z-\omega}$$

Take $B = 0$ for the sake of simplicity, and consider the sum of terms

$$\sum_{\omega} \frac{1}{(z-\omega)^2}$$

summing over $\omega = m\omega_1 + n\omega_2$ with $z \neq \omega$. Moving forward in this section, denote the set of all ω as Ω.

Lemma 6.1. If α is real, the infinite series

$$\sum_{\omega\in\Omega,\omega\neq 0} \frac{1}{\omega^\alpha}$$

converges absolutely if and only if $\alpha > 2$.

Proof. First, see Figure 2, which depicts four period parallelograms from the lattice generated by the periods ω_1 and ω_2 along with with the minimum and maximum distances from the origin to the parallelogram.

Referring to the aforementioned figure, we denote r as the minimum distance from the origin to the parallelogram immediately around the origin and R as the maximum distance from the origin to said parallelogram. Now, we have $r \leq |\omega| \leq R$ for the 8 periods ω in the figure. The next layer of periods will give us 16 new periods, so we will have $2r \leq |w| \leq 2R$. This trend continues as each new layer of periods is added.

Now, in an attempt to bound the sum that was stated in the lemma, we invert our trend from above so that we have $\frac{1}{R^\alpha} \leq \frac{1}{|\omega|^\alpha} \leq \frac{1}{r^\alpha}$ for the first 8 periods about the origin. Then, similarly to before, we have $\frac{1}{(2R)^\alpha} \leq \frac{1}{|\omega|^\alpha} \leq \frac{1}{(2r)^\alpha}$ for the additional 16 periods in the next layer about the origin. This trend continues for each new concentric layer of periods about the origin. Thus, for the sum $S(n) = \sum \frac{1}{|\omega|^\alpha}$ over the $8(1 + ... + n)$ periods in concentric layers about the origin, we have the following:

$$\frac{8}{R^\alpha} + ... + \frac{8n}{(nR)^\alpha} \leq S(n) \leq \frac{8}{r^\alpha} + ... + \frac{8n}{(nr)^\alpha},$$

or

$$\frac{8}{R^\alpha}\sum_{k=1}^{n}\frac{1}{k^{\alpha-1}} \leq S(n) \leq \frac{8}{r^\alpha}\sum_{k=1}^{n}\frac{1}{k^{\alpha-1}}.$$

Now we can see that the partial sums of the form $S(n)$ are bounded above if $\alpha > 2$. Thus, the series $\sum_{\omega\in\Omega} \frac{1}{\omega^\alpha}$

converges if $\alpha > 2$. It can be seen as well that the lower bound would be divergent for any $\alpha \leq 2$. □

Lemma 6.2. If $\alpha > 2$ and $R > 0$, the series

$$\sum_{|\omega|>R} \frac{1}{(z-\omega)^\alpha}$$

converges absolutely and uniformly in the disk $|z| \leq R$.

Proof. Refer again to Figure 2.

We want to show that there is a constant M that depends on R and α so, for $\alpha \geq 1$, we have

$$\frac{1}{|z-\omega|^\alpha} \leq \frac{M}{|\omega|^\alpha}$$

for all ω with $|\omega| > R$ and for all z with $|z| \leq R$. We can see that the previous inequality is the same as

$$\left|\frac{z-\omega}{\omega}\right|^\alpha \geq \frac{1}{M}.$$

Now, we aim to explicitly find M. Consider $\omega \in \Omega$ with $|\omega| > R$ and choose an ω whose norm is minimal. Denote this as $|\omega| = R + d$ for some $d > 0$. Then, for $|z| \leq R$, and the period denoted above, we have the following:

$$\left|\frac{z-\omega}{\omega}\right| = \left|1 - \frac{z}{\omega}\right| \geq 1 - \left|\frac{z}{w}\right| \geq 1 - \frac{R}{R+d}.$$

Thus,

$$\left|\frac{z-\omega}{\omega}\right|^\alpha \geq \left(1 - \frac{R}{R+d}\right)^\alpha = \frac{1}{M}$$

so

$$M = \left(1 - \frac{R}{R+d}\right)^{-\alpha}.$$

Following the procedure laid out in Lemma 6.1 gives us the desired result. □

Theorem 6.3. Let f be defined by the series

$$f(z) = \sum_{\omega\in\Omega} \frac{1}{(z-\omega)^3}.$$

Then f is an elliptic function with periods ω_1 and ω_2 and with a pole of order 3 at each period in Ω.

Proof. To show that f is doubly periodic, we will show that it has two periods ω_1 and ω_2. We must show that $f(z+\omega_1) = f(z)$ and $f(z+\omega_2) = f(z)$. We have

$$f(z+\omega_1) = \sum_{\omega\in\Omega} \frac{1}{(z+\omega_1-\omega)^3}.$$

Since $\omega - \omega_1$ runs through each period in Ω along with ω, $f(z+\omega_1)$ is only a rearrangement of the series for $f(z)$. Then, by absolute convergence of $f(z)$, we have that $f(z+\omega_1) = f(z)$. The same argument can be made for ω_2, so f is doubly periodic.

All that remains is to show that f is meromorphic. By Lemma 6.2, summing the series $f(z)$ over the periods $|\omega| > R$ leads to uniform convergence in the disk $|z| \leq R$, so it represents an analytic function within the given disk. The finite number of remaining terms are analytic as well, except for a pole of order 3 at each period ω in the disk. Thus, f is meromorphic with a pole order 3 at each period ω. Now, we have that f is elliptic. □

This is our first example of an elliptic function. The stricter conditions make it much more difficult than considering the case of periodic functions. It was mentioned in Section 5, moreover, that elliptic functions must have order $n \geq 2$. This section explored the creation of an elliptic function of order 3. The next sections will explore the creation of an elliptic function of order 2 and the properties of such a function.

7 The Weierstrass $\wp$ function

An elliptic function of order 2 is constructed via term-by-term integration of the function defined in Theorem 6.3. Said function is called the Weierstrass $\wp$ function.

Definition 7.1. The **Weierstrass $\wp$ function** is defined by the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}.$$

Theorem 7.1. The function $\wp$ has periods ω_1 and ω_2. It is analytic except for a double pole at each period $\omega \in \Omega$. Furthermore, $\wp(z)$ is an even function of z.

Proof. Every term in the series of $\wp(z)$ has the norm

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{\omega^2 - (z-\omega)^2}{\omega^2 (z-\omega)^2} \right| = \left| \frac{z(2\omega - z)}{\omega^2 (z-\omega)^2} \right|.$$

Now, consider a compact disk $|z| \leq R$. This disk only contains a finite number of periods ω. Excluding the terms of the series that contains the periods in the disk, we are left with

$$\left| \frac{1}{(z-\omega)^2} \right| \leq \frac{M}{|\omega|^2}$$

from the proof of Lemma 6.2, where M is a constant that depends on R. Then, we have the following since $R < |\omega|$ for ω outside of the disk $|z| \leq R$:

$$\left| \frac{z(2\omega - z)}{\omega^2 (z-\omega)^2} \right| \leq \frac{MR(2|\omega| + R)}{|\omega|^4} \leq \frac{MR(2 + R/|w|)}{|\omega|^3} \leq \frac{3MR}{|\omega|^3}$$

We truncated the series since we excluded terms in the series as previously directed. We have bounded this truncated series, so we have shown that it converges absolutely and uniformly in the disk $|z| \leq R$. Thus, this series is analytic in this disk. Meanwhile, the excluded terms give a second order pole at ω inside the disk, so $\wp$ is analytic except for a double pole at each period.

Next, we wish to show that $\wp$ is an even function. To do this, notice that

$$(-z-\omega)^2 = (z+\omega)^2 = (z-(-\omega))^2.$$

Given that $-\omega$ runs through all non-zero periods with ω, we have that $\wp(-z) = \wp(z)$, so $\wp$ is even. □

Consider the following propositions to uncover more properties about the Weierstrass $\wp$ function:

Proposition 7.2. $\wp(u) = \wp(v)$ if $u - v$ or $u + v$ is a period of $\wp$.

Proof. Assume that $u - v$ or $u + v$ is a period of $\wp$.

If $u - v$ is a period, $\wp(v + (u - v)) = \wp(v)$. Otherwise, if $u + v$ is a period, $-(u + v)$ must also be a period by definition. Then, $\wp(u - (u + v)) = \wp(u)$, so $\wp(-v) = \wp(u)$, which implies that $\wp(v) = \wp(u)$ since $\wp$ is an even function. □

Proposition 7.3. Let $a_1, ..., a_n$ and $b_1, ..., b_m$ be complex numbers such that none of the numbers $\wp(a_i) - \wp(b_j)$ equals 0. Let $f(z) = \prod_{k=1}^{n}[\wp(z) - \wp(a_k)]/\prod_{r=1}^{m}[\wp(z) - \wp(b_r)]$. Then f is an even elliptic function with zeros at $a_1, ..., a_n$ and poles at $b_1, ..., b_m$.

Proof. We have that f is even since $\wp$ is even (Theorem 7.1). It is clear that $f(z) = f(-z)$ with this fact. It is clear that f is elliptic by construction since $\wp$ is elliptic. Furthermore, setting $z = a_i$ when $i = k$, $\wp(z) - \wp(a_k) = 0$ and $\wp(a_i) - \wp(b_r) \neq 0$. By definition, $a_1, ..., a_n$ are zeros of f. Bearing in mind that poles, by definition, make the denominator of a complex function zero, a similar argument shows that $b_1, ..., b_m$ are poles of f. □

8 The Laurent expansion of $\wp$ near the origin

Theorem 8.1. Let $r = \min\{|\omega| : \omega \neq 0\}$. Then for $0 < |z| < r$, we have

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{2n+2}z^{2n}$$

where

$$G_n = \sum_{\omega \neq 0} \frac{1}{\omega^n}$$

for $n \geq 3$.

Proof. If $0 < |z| < r$, then $\left|\frac{z}{\omega}\right| < 1$, so we have

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2\left(1 - \frac{z}{\omega}\right)^2} = \frac{1}{\omega^2}\left(1 + \sum_{n=1}^{\infty}(n+1)\left(\frac{z}{\omega}\right)^n\right).$$

This was obtained by considering a geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and then differentiating both sides to obtain $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$.

Now, we have

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \sum_{n=1}^{\infty} \frac{n+1}{\omega^{n+2}} z^n.$$

Summing over all ω, we can see that

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)\sum_{\omega \neq 0}\frac{1}{\omega^{n+2}}z^n = \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)G_{n+2}z^n$$

for G_n as previously defined. Since $\wp(z)$ is even, furthermore, the coefficients G_{2n+1} (the odd coefficients) must disappear, so we are left with $\wp(z)$ as defined in the statement of the theorem. □

9 Differential equation satisfied by $\wp$

Theorem 9.1. The function $\wp$ satisfies the nonlinear differential equation

$$[\wp'(z)]^2 = 4\wp^3(z) - 60G_4\wp(z) - 140G_6.$$

Proof. Take a linear combination of powers of $\wp$ and $\wp'$ that eliminates the pole at $z = 0$. The result is an elliptic function with no poles, so it must be a constant function. To accomplish this, consider a power series expansion of $\wp'$ near $z = 0$. Now, we have

$$\wp'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \cdots$$

which is an elliptic function with order 3. Since the equation we are after has the square of $\wp'$, consider

$$[\wp'(z)]^2 = \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + \cdots$$

Note that this represents a power series in z that vanishes at $z = 0$. Next, consider

$$4\wp^3(z) = \frac{4}{z^6} + \frac{36G_4}{z^2} + 60G_6 + \cdots$$

so

$$[\wp'(z)]^2 - 4\wp^3(z) = -\frac{60G_4}{z^2} - 140G_6 + \cdots$$

and

$$[\wp'(z)]^2 - 4\wp^3(z) + 60G_4\wp(z) = -140G_6 + \cdots$$

Now the left-hand side has no pole at $z = 0$ by construction, so it must be constant. Thus, any non-constant terms on the right-hand side must disappear, so the constant is $-140G_6$. This gives us the desired result. □

10 The Eisenstein series and the invariants g_2 and g_3

Definition 10.1. If $n \geq 3$, the series

$$G_n = \sum_{\omega \neq 0} \frac{1}{\omega^n}$$

as defined in Theorem 8.1 is called the **Eisenstein series of order** n. The **invariants** g_2 **and** g_3 are the numbers defined by the relations $g_2 = 60G_4$ and $g_3 = 140G_6$.

The differential equation from the previous section only utilizes g_2 and g_3, so $\wp$ ought to be determined only by g_2 and g_3. The coefficients G_n can also be determined only from g_2 and g_3, as shown in the following theorem.

Theorem 10.1. Each Eisenstein series G_n is expressible as a polynomial in g_2 and g_3 with positive rational coefficients. Actually, if $b(n) = (2n+1)G_{2n+2}$, we have the recursion relations

$$b(1) = \frac{g_2}{20}, b(2) = \frac{g_3}{28}$$

and

$$(2n+3)(n-2)b(n) = 3\sum_{k=1}^{n-2} b(k)b(n-1-k)$$

for $n \geq 3$. Equivalently,

$$(2m+1)(m-3)(2m-1)G_{2m} = 3\sum_{r=2}^{m-2}(2r-1)(2m-2r-1)G_{2r}G_{2m-2r}$$

for $m \geq 4$.

Proof. If we differentiate the differential equation for $\wp$ from the previous section, we obtain another differential equation of order 2 satisfied by $\wp$:

$$\wp''(z) = 6\wp^2(z) - \frac{1}{2}g_2.$$

Now, consider $\wp(z) = z^{-2} + \sum_{n=1}^{\infty} b(n)z^{2n}$. Then, equate the like powers of z in $\wp''(z) = 6\wp^2(z) - \frac{1}{2}g_2$ to obtain the recursion relations in the statement of the theorem. □

11 The numbers e_1, e_2, and e_3

Definition 11.1. We define e_1, e_2, and e_3 as the values of $\wp$ at the half-periods

$$e_1 = \wp\left(\frac{\omega_1}{2}\right), e_2 = \wp\left(\frac{\omega_2}{2}\right), e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$$

In the next theorem, we will see that these numbers are the roots of the polynomial $4\wp^3 - g_2\wp - g_3$.

Theorem 11.1. We have

$$4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp^3(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Furthermore, the roots of e_1, e_2, and e_3 are distinct, so $g_2^3 - 27g_3^2 \neq 0$.

Proof. We know that $\wp$ is an even function, so that tells us that the derivative $\wp'$ is odd. To approach the desired result, we will show that the half-periods of an odd elliptic function are either zeros or poles. By periodicity of $\wp$ and $\wp'$, we have that $\wp'(-\frac{1}{2}\omega) = \wp'(\omega - \frac{1}{2}\omega) = \wp'(\frac{1}{2}\omega)$. Furthermore, since $\wp'$ is odd, $\wp'(-\frac{1}{2}\omega) = -\wp'(\frac{1}{2}\omega)$, so $\wp'(\frac{1}{2}\omega) = 0$ if $\wp'(\frac{1}{2}\omega)$ is finite.

We know that $\wp'$ has no poles at $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, or $\frac{1}{2}(\omega_1+\omega_2)$, so these points must be zeros of $\wp'$. They must be simple zeros, however, since the order of $\wp'$ is 3. Thus, this differential equation shows that each of the aforementioned points, $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, and $\frac{1}{2}(\omega_1+\omega_2)$, are also zeros of the cubic function in the statement of the theorem. Now, we have the desired factorization with the zeros e_1, e_2, and e_3 defined as $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, and $\frac{1}{2}(\omega_1+\omega_2)$, respectively.

Moving forward, we want to show that these roots are distinct. Since $\wp(z) - e_1$ vanishes at $z = \frac{1}{2}\omega_1$, we have a double zero since $\wp'(\frac{1}{2}\omega_1) = 0$ as well. Similarly, there is a double zero for $\wp(z) - e_2$ at $z = \frac{1}{2}\omega_2$ associated with e_2. If $e_1 = e_2$, we would have double zeros both at $\frac{1}{2}\omega_1$ and $\frac{1}{2}\omega_2$, so the order of the function would be at least 4. However, the order of $\wp$ is 2 by construction, so this is a contradiction. Thus, e_1 and e_2 are distinct. Analogous arguments can show that $e_1 \neq e_3$ and $e_2 \neq e_3$.

Finally, if a polynomial has distinct roots, its discriminant does not vanish. The discriminant of the cubic polynomial $4x^3 - g_2x - g_3$ was shown, with outside computation, to be $g_2^3 - 27g_3^2$. Set $x = \wp(z)$ and the roots of the polynomial are distinct, so $g_2^3 - 27g_3^2 \neq 0$. □

12 The discriminant Δ

Definition 12.1. The **discriminant** is the number $\Delta = g_2^3 - 27g_3^2$. Moreover, g_2, g_3, and Δ are all considered as functions of the periods ω_1 and ω_2:

$$g_2 = g_2(\omega_1, \omega_2), g_3 = g_3(\omega_1, \omega_2), \Delta = \Delta(\omega_1, \omega_2)$$

From the Eisenstein series, furthermore, we can see that g_2 and g_3 are of degrees -4 and -6 (orders 4 and 6), respectively, so $g_2(\lambda\omega_1, \lambda\omega_2) = \lambda^{-4}g_2(\omega_1, \omega_2)$ and $g_3(\lambda\omega_1, \lambda\omega_2) = \lambda^{-6}g_3(\omega_1, \omega_2)$ for some non-zero λ. Thus, we can see based on the definition of the discriminant that the degree of Δ is -12, so $\Delta(\lambda\omega_1, \lambda\omega_2) = \lambda^{-12}\Delta(\omega_1, \omega_2)$.

In order to make these functions of one variable, we usher in a change of variables: let $\lambda = \frac{1}{\omega_1}$ and $\tau = \frac{\omega_2}{\omega_1}$. Then, we have

$$g_2(1, \tau) = \omega_1^4 g_2(\omega_1, \omega_2), g_3(1, \tau) = \omega_1^6 g_3(\omega_1, \omega_2), \Delta(1, \tau) = \omega_1^{12}\Delta(\omega_1, \omega_2)$$

Now we have these as functions of one complex variable τ. Moreover, we can label ω_1 and ω_2 in such a way that τ, as defined, will have a positive imaginary component. Then, we can restrict our study of these functions to the upper half-plane $H = \text{Im}(\tau) > 0$. Now, if we denote $(1, \tau)$ by τ, we obtain the following:

$$g_2(\tau) = 60 \sum_{m,n=-\infty,(m,n)\neq 0}^{+\infty} \frac{1}{(m + n\tau)^4}$$

$$g_3(\tau) = 140 \sum_{m,n=-\infty,(m,n)\neq 0}^{+\infty} \frac{1}{(m + n\tau)^6}$$

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$$

Note that Theorem 11.1 guarantees that $\Delta(\tau) \neq 0$ for all $\tau \in H$.

13 Klein's modular function $J(\tau)$

Definition 13.1. If the ratio $\frac{\omega_2}{\omega_1}$ is not real, we define **Klein's modular function** J as

$$J(\omega_1, \omega_2) = \frac{g_2^3(\omega_1, \omega_2)}{\Delta(\omega_1, \omega_2)}$$

Since g_2^3 and Δ are homogeneous and have the same degree, J is homogeneous of degree zero, so $J(\lambda\omega_1, \lambda\omega_2) = J(\omega_1, \omega_2)$ by definition. Now, in a similar fashion to the previous section, we can write $J(\tau)$ as a function of one complex variable studied in the upper half-plane H.

Theorem 13.1. The functions $g_2(\tau)$, $g_3(\tau)$, $\Delta(\tau)$, and $J(\tau)$ are all analytic in H.

We will consider the idea of the proof. Since we know from the previous section that $\Delta(\tau) \neq 0$ in H, we need only show that g_2 and g_3 are analytic in H. We know from previous sections that g_2 and g_3 are represented by series of the form

$$\sum_{m,n=-\infty;(m,n)\neq(0,0)}^{+\infty} \frac{1}{(m+n\tau)^{\alpha}}$$

where m and n are integers and $\alpha > 2$. To show that these are analytic, fix τ in H. It must be shown that the series above converges absolutely in H and uniformly in any strip $S = \{\tau = x + iy : |x| \leq A, y \geq \delta > 0\}$. Now, all that remains is to show that $\frac{1}{|m+n\tau|^{\alpha}}$ is bounded by some $M > 0$ that only depends on A and δ.

14 Invariance of J under unimodular transformations

Definition 14.1. Consider two given periods ω_1 and ω_2 with a non-real ratio, as is customary. Now, we define ω_1' and ω_2' by

$$\omega_2' = a\omega_2 + b\omega_1$$
$$\omega_1' = c\omega_2 + d\omega_1$$

where a, b, c, and d are integers such that $ad - bc = 1$.

Since ω_1' and ω_2' are linear combinations of ω_1 and ω_2, (ω_1', ω_2') is equivalent to (ω_1, ω_2). In other words, both pairs generate the same lattice of periods. Consequently, $g_2(\omega_1, \omega_2) = g_2(\omega_1', \omega_2')$ and $g_3(\omega_1, \omega_2) = g_3(\omega_1', \omega_2')$. Since Δ and J are both defined in terms of g_2 and g_3, furthermore, $\Delta(\omega_1', \omega_2') = \Delta(\omega_1, \omega_2)$ and $J(\omega_1', \omega_2') = J(\omega_1, \omega_2)$.

Now, consider the new ratio of periods

$$\tau' = \frac{\omega_2'}{\omega_1'} = \frac{a\omega_2 + b\omega_1}{c\omega_2 + d\omega_1} = \frac{a\tau + b}{c\tau + d}$$

with $\tau = \frac{\omega_2}{\omega_1}$ as before. Moreover, $\text{Im}(\tau') = \text{Im}(\frac{a\tau+b}{c\tau+d}) = \frac{ad-bc}{|c\tau+d|^2}\text{Im}(\tau) = \text{Im}(\tau)\frac{1}{|c\tau+d|^2}$ since $ad - bc = 1$. Thus, τ' is in H if and only if τ is in H.

Definition 14.2. The equation

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

with integers a, b, c, d such that $ad - bc = 1$ is a **unimodular transformation**. The **modular group** is the set of all unimodular transformations that forms a group under composition.

Now that we have defined unimodular transformations, we can get to the subject of this section: the invariance of $J(\tau)$ under unimodular transformations. Consider the following two theorems:

Theorem 14.1. If τ is in H and a, b, c, d are integers such that $ad - bc = 1$, then $\frac{a\tau+b}{c\tau+d}$ is in H and

$$J\left(\frac{a\tau + b}{c\tau + d}\right) = J(\tau).$$

Consider a particular unimodular transformation $\tau' = \tau + 1$. The above equation shows that $J(\tau') = J(\tau)$, so J is a periodic function of τ.

Theorem 14.2. If τ is in H, $J(\tau)$ can be defined by an absolutely convergent Fourier series

$$J(\tau) = \sum_{n=-\infty}^{\infty} a(n)e^{2\pi i n\tau}$$

Proof. Consider the change of variable $x = e^{2\pi i\tau}$. This maps H into the punctured disk $D = \{x : 0 < |x| < 1\}$ (see figure to be inserted later). Each τ in H maps to a unique point x in D and each x is the image of infinitely many points from H. Now, if τ and τ' both map to x in D, we have $e^{2\pi i\tau} = e^{2\pi i\tau'}$, so τ and τ' only differ by an integer.

Now, let $f(x) = J(\tau)$. This function f is well-defined: since J is periodic with period 1, as seen in the previous theorem, it has the same value at all of the points τ. Furthermore, f is analytic in D since

$$f'(x) = \frac{d}{dx}J(\tau) = \frac{d}{d\tau}J(\tau)\frac{d\tau}{dx} = \frac{J'(\tau)}{dx/d\tau} = \frac{J'(\tau)}{2\pi i e^{2\pi i\tau}}$$

so $f'(x)$ exists for every point in D. Now, since f is analytic in the disk D, it has a Laurent expansion about 0 in D, so

$$f(x) = \sum_{n=-\infty}^{\infty} a(n)x^n.$$

This converges absolutely for each point x in D. Reinstating our change of variable $x = e^{2\pi i\tau}$, we achieve the desired Fourier series representation for $J(\tau)$. □

15 The Fourier expansions of $g_2(\tau)$ and $g_3(\tau)$

The focus of this section is to determine the Fourier coefficients of g_2 and g_3. Recall,

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4}$$
$$g_3 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^6}$$

These are Eisenstein series of orders 4 and 6, respectively, by definition. This comes from the Laurent expansion of $\wp$. These are double series, so it is useful to start by considering the Fourier expansions of $\sum_{m=-\infty}^{\infty} \frac{1}{(m+n\tau)^4}$ and $\sum_{m=-\infty}^{\infty} \frac{1}{(m+n\tau)^6}$. More specifically, consider the following results.

Lemma 15.1. If τ is an element of H and $n > 0$, we have the Fourier expansions

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+n\tau)^4} = \frac{8\pi^4}{3} \sum_{r=1}^{\infty} r^3 e^{2\pi i r n \tau}$$
$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+n\tau)^6} = -\frac{8\pi^6}{15} \sum_{r=1}^{\infty} r^5 e^{2\pi i r n \tau}$$

This is achieved by considering first the partial fraction decomposition of $\pi \cot(\pi\tau)$. Then, substituting $x = e^{2\pi i \tau}$ and differentiating repeatedly to find the Fourier coefficients, we arrive at the conclusion of the lemma. Now, all that remains is to consider the final result from this section. For the sake of brevity, we will overlook the proof of this theorem.

Theorem 15.2. If τ is an element of H, we have the Fourier expansions

$$g_2(\tau) = \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) e^{2\pi i k \tau} \right\}$$
$$g_3(\tau) = \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k) e^{2\pi i k \tau} \right\}$$

with $\sigma_\alpha(k) = \sum_{d|k} d^\alpha$.

With these results, we can see the explicit Fourier expansions of g_2 and g_3.

16 The Fourier expansions of $\Delta(\tau)$ and $J(\tau)$

In this section, the explicit Fourier expansions of $\Delta(\tau)$ and $J(\tau)$ will be stated. These Fourier expansions are determined using the functions $\tau(n)$, Ramanujan's tau function, and $c(n)$, which produces integers. Both will be listed in the index of figures more concretely. Now, for the Fourier expansion of $\Delta(\tau)$, consider the following:

Theorem 16.1. If τ is an element of H, we have the Fourier expansion

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n \tau}.$$

Furthermore, for the Fourier expansion of $J(\tau)$, consider the following:

Theorem 16.2. If τ is an element of H, we have the Fourier expansion

$$12^3 J(\tau) = e^{-2\pi i \tau} + 744 + \sum_{n=1}^{\infty} c(n) e^{2\pi i n \tau}$$

where the $c(n)$ are integers.

17 Möbius transformations

We discussed unimodular transformations in previous sections. In this section, we will discuss more general transformations: Möbius transformations.

Definition 17.1. A **Möbius transformation** is defined by

$$f(z) = \frac{az + b}{cz + d}$$

for arbitrary complex numbers a, b, c, d where $ad - bc \neq 0$.

This f is defined for all z in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ with the exception of $z = -\frac{d}{c}$ and $z = \infty$. To remedy these exceptions and extend the definition of f to the remainder of $\mathbb{C}^*$, let $f\big(-\frac{d}{c}\big) = \infty$ and $f(\infty) = \frac{a}{c}$. This follows from the usual understanding from calculus that $\frac{z}{0} = \infty$ when $z \neq 0$. Furthermore, we must have the restriction $ad - bc \neq 0$. To see this, consider

$$f(w) - f(z) = \frac{(ad - bc)(w - z)}{(cw + d)(cz + d)}.$$

If $ad - bc = 0$, f would be a constant function, which is a degenerate case. Also, this transformation is analytic on $\mathbb{C}^*$ except for a simple pole at $z = -\frac{d}{c}$. Note, then, that these transformations are not elliptic since elliptic functions must have order greater than or equal to 2.

If we solve for z in the definition of a Möbius transformation, we obtain

$$z = \frac{df(z) - b}{-cf(z) + a}$$

which shows that f maps $\mathbb{C}^*$ onto $\mathbb{C}^*$. This shows, too, that the inverse f^{-1} is a Möbius transformation. Moreover, the equation for $f(w) - f(z)$ shows that f is one-to-one on $\mathbb{C}^*$ since it is non-zero. The only way for this equation to equal 0 would be for w to equal z.

Now, evaluating $\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$ gives us

$$f'(z) = \frac{ad - bc}{(cz + d)^2}$$

which shows that $f'(z) \neq 0$ for each point of analyticity. This means that f is conformal everywhere except for possibly at the pole $z = -\frac{d}{c}$. Note that a conformal map preserves local angles and orientation of figures but not necessarily size or curvature of figures.

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